

On Partial Maximally-Recoverable and Maximally-Recoverable Codes

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Abstract—An $[n, k]$ linear code \mathcal{C} that is subject to locality constraints imposed by a parity check matrix H_0 is said to be a maximally recoverable (MR) code if it can recover from any erasure pattern that some k -dimensional subcode of the null space of H_0 can recover from. The focus in this paper is on MR codes constrained to have all-symbol locality r . Given that it is challenging to construct MR codes having small field size, we present results in two directions. In the first, we relax the MR constraint and require only that apart from the requirement of being an optimum all-symbol locality code, the code must yield an MDS code when punctured in a single, specific pattern which ensures that each local code is punctured in precisely one coordinate and that no two local codes share the same punctured coordinate. We term these codes as partially maximally recoverable (PMR) codes. We provide a simple construction for high-rate PMR codes and then provide a general, promising approach that needs further investigation. In the second direction, we present three constructions of MR codes with improved parameters, primarily the size of the finite field employed in the construction.

Index Terms—Distributed storage, codes with locality, maximally recoverable codes, partial-MDS codes.

I. INTRODUCTION

In a distributed storage network, each file is regarded as a message, encoded into a codeword by adding redundancy, and stored in the network. Each code symbol is typically placed on a different node to provide resiliency against node failure. Both replication and Reed-Solomon codes are commonly employed to protect data but have their drawbacks. While replication incurs large overhead, RS codes are inefficient when it comes to node repair. The notion of codes with locality introduced in [1], was motivated in part, by this shortcoming of an RS code.

A. Codes with Locality

Definition 1: [1] An $[n, k]$ code \mathcal{C} of block length n and dimension k is said to have all-symbol locality r if for every code symbol c_i in \mathcal{C} , the dual code \mathcal{C}^\perp contains a codeword with support L_i satisfying $i \in L_i$ and $|L_i| \leq (r + 1)$. We will call L_i the recovery set for code symbol i . We assume w.l.o.g. that $L_i \not\subset \cup_{j \in [n], j \neq i} L_j$. We will write $[n, k]_r$ to indicate an $[n, k]$ code with such all-symbol locality r and $[n, k, d]_r$ if the code has minimum distance d .

Codes with all-symbol locality have the property that the number of code symbols that need to be accessed to repair a failed node is at most r . The following bound on the minimum

distance under a weaker notion called information-symbol locality was derived in [1]:

$$d_{\min} \leq (n - k + 1) - \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right). \quad (1)$$

The same bound also applies to codes with all-symbol locality and is often (but not always) tight, see [2] for instance. The Pyramid codes introduced in [3] are shown in [1] to be an example of codes with information-symbol locality that are optimal with respect to this bound. The existence of code with all-symbol locality was established in [1] for the case when $(r + 1) \mid n$. Codes with locality also go by the names locally repairable codes [4] or local reconstruction codes [5].

A class of codes with all-symbol locality known as *homomorphic self-repairing codes* were constructed in [6] with the aid of linearized polynomials. An example provided in [6] is optimal with respect to the bound in (1). A general construction of optimal codes with all-symbol locality is provided in [7], that is based on the construction of Gabidulin maximum rank-distance codes. An upper bound on minimum distance, similar to that in (1), was derived in [4], that applies also to non-linear codes. Also provided, in [4], is an explicit construction of a class of linear, optimal all-symbol locality codes possessing a vector alphabet. This construction is related to an earlier construction in [8], of codes termed as simple regenerating codes. Most recently, Tamo and Barg [9] have provided general constructions for optimal codes with all-symbol locality.

B. Maximally Recoverable Codes

The notion of a maximally recoverable code is most easily defined in terms of the generator matrix G of the code.

Let \mathcal{C} be an $[n, k]_r$ code that satisfies the all-symbol, locality- r constraints imposed by a parity-check matrix H_0 . Let \mathcal{C}_0 denote the null space of H_0 and G_0 be the corresponding generator matrix. Then \mathcal{C} is said to be an MR code with respect to H_0 if for any collection of k linearly independent columns in G_0 , the corresponding columns of G are also linearly independent.

The construction of optimum codes with locality given in [9], has field size on the order of block length. A principal code constructed in their paper corresponds to a subcode of an RS code. The coordinates of this code are grouped together in accordance with cosets of a cyclic subgroup of

the group of n th roots of unity. The subcode of the RS code is selected so that the restriction of the RS code to a coset of size $(r + 1)$ corresponds to evaluation of a polynomial of degree $(r - 1)$, thus providing locality. The degree of the encoding polynomials is shown to be such that the resulting codes are optimal with respect to the minimum distance bound in (1). The authors in [10] define a general notion of maximal recoverable codes and provide a construction for maximally recoverable codes of field size $\binom{n-1}{k-1}$. In [11], a general form of parity-check matrix was considered with the aim of constructing MR codes. These codes are referred to in [11] as partial MDS codes. The authors provide conditions under which the proposed form of parity-check matrix defines an MR code and identify explicit parameter sets for which their construction results in an MR code. A particular instance of their construction has field size $O(2^n)$, where n is the block length of the code. For the case of a single global parity check, the authors provide a construction where the field size is $O(n)$.

The authors of [12], construct codes termed as sector-disk (SD) codes. These are codes which for certain puncturing patterns associated to a combination of disk and sector failures result in MDS codes. The authors provide a construction for the case of 2 global parities for handling the correction of a single or double erasure in each local code and present a parameter range for which their construction satisfies the requirement of an SD code through computer search. In [13], the authors present a construction for maximally recoverable codes with 2 global parities with field size of $O(n)$ that can handle single erasures through local error correction. In [14], a construction of SD codes with 2 global parities is provided having field size of $O(n)$ to handle one or two erasures in each local code. This was subsequently strengthened in [15], where a construction of SD code and partial MDS code was provided for 2 global parities having field size of $O(n)$ that can handle any number of erasures through local error correction.

In [16], a family of explicit, MR codes for single local erasure correction is provided in which the number of global parities can be arbitrary. It is assumed here that $(r + 1) \mid n$ where r is the locality parameter of the code. The parity check matrix in [16] has the same form as in [11] except that the authors use variables to fill up the entries of the parity check matrix and then proceed to derive conditions needed to be satisfied by these variables in order to yield an MR code.

In [17], a relaxation in the definition of an MR code is proposed. Here the authors seek to correct a select set of erasure patterns. Each codeword is put into matrix form in such a way that each row corresponds to a local code. A vector is used to specify the number of columns of this code matrix in which erasure can occur, the maximum number of erasures allowed within each column as well as the maximum number of complete column erasures permitted. A construction satisfying these requirements is provided.

In the present paper, a relaxation of the MR criterion termed as a partial maximally recoverable (PMR) criterion is presented and a simple, high-rate construction provided. Also contained in the paper are three constructions of MR codes

with improved parameters, primarily field size.

II. PARTIAL MAXIMUM RECOVERABILITY

Given that the construction of MR codes having small field size is challenging, we seek here to construct codes that satisfy a weaker condition which we will refer to in this paper as the partial maximally recoverable (PMR) condition. Let \mathcal{C} be an $[n, k]_r$ code having all-symbol locality and whose minimum distance satisfies the bound in (1) with equality. Let L_i denote the recovery sets. In the context of PMR codes, an admissible puncturing pattern $\{e_1, e_2, \dots, e_m\}$ is one in which the $\{e_i\}$ satisfy the condition:

$$e_i \in L_i \setminus \left(\bigcup_{j \in [m], j \neq i} L_j \right).$$

A PMR code is then defined simply as an optimal all-symbol locality code which becomes an MDS code upon puncturing under some admissible puncturing pattern. The parity-check matrix of a PMR code is characterized below. We assume w.l.o.g. in the section below, that $\{e_1, e_2, \dots, e_m\} = (1, 2, \dots, m)$ through symbol reordering.

A. Characterizing H for a PMR Code

Theorem 2.1: Let \mathcal{C} be a PMR code as defined above for admissible puncturing pattern $e = \{e_1, \dots, e_m\}$. Then \mathcal{C} can be assumed to have parity-check matrix of the form:

$$H = \left[\begin{array}{c|c} I_m & \underbrace{F}_{(m \times k_0)} \\ \hline [0] & \underbrace{H_{\text{MDS}}}_{(\Delta \times k_0)} \end{array} \right],$$

where H_{MDS} is the parity-check matrix of an $[k_0, k_0 - \Delta]$ MDS code and F is of the form:

$$F = \begin{bmatrix} \underline{x}_1^t \\ \underline{x}_2^t \\ \vdots \\ \underline{x}_m^t \end{bmatrix}$$

in which each \underline{x}_i is a vector of Hamming weight at most r .

Proof: Clearly, H can be assumed to be of the form

$$H = \left[\begin{array}{c|c} I_m & \underbrace{F}_{(m \times k_0)} \\ \hline H_1 & \underbrace{H_2}_{(\Delta \times k_0)} \end{array} \right],$$

which can be transformed, upon row reduction to the form:

$$H = \left[\begin{array}{c|c} I_m & \underbrace{F}_{(m \times k_0)} \\ \hline [0] & \underbrace{H_3}_{(\Delta \times k_0)} \end{array} \right].$$

It is desired that upon puncturing the first m coordinates (corresponding to coordinates of the identity matrix I_m in the upper left), the code be MDS. But since the dual of a

punctured code is the shortened code in the same coordinates, it follows that H_3 must be the parity-check matrix of an MDS code. ■

B. A Simple Parity-Splitting Construction for a PMR Code when $\Delta \leq (r-1)$

We will assume throughout the rest of the paper that C is an $[n, k]_r$ code where $(r+1)|n$ and having parameters m, Δ given by:

$$\begin{aligned} n &= m(r+1), & k_0 &= mr, \\ k &= k_0 - \Delta = n - (m + \Delta). \end{aligned}$$

Thus Δ represents the number of “global” parity checks imposed on top of the m “local” parity checks.

Assume that $\Delta \leq (r-1)$. Let H_0 be the the $(\Delta+1 \times k_0)$ parity-check matrix of an MDS code. Let \underline{x}^t be the last row of H_0 and H_1 be H_0 with the last row deleted, i.e.,

$$H_0 = \begin{bmatrix} H_1 \\ \underline{x}^t \end{bmatrix}.$$

In the construction, we will require that H_1 also be the parity-check matrix of an MDS code and set $H_{\text{MDS}} = H_1$. For example, this is the case when H_0 is either a Cauchy or a Vandermonde matrix. Let $\{\underline{x}_i^t\}_{i=1}^m$ be the m contiguous component $(1 \times r)$ vectors of \underline{x}^t defined through

$$\underline{x}^t = (\underline{x}_1^t \ \underline{x}_2^t \ \cdots \ \underline{x}_m^t).$$

Let F be given by

$$F = \begin{bmatrix} \underline{x}_1^t & & & \\ & \underline{x}_2^t & & \\ & & \ddots & \\ & & & \underline{x}_m^t \end{bmatrix}.$$

Lemma 2.2:

$$\lceil \frac{mr - \Delta}{r} \rceil = m - \lfloor \frac{\Delta}{r} \rfloor.$$

Theorem 2.3 (Parity-Splitting Construction): The $[n, k]$ code C having parity-check matrix H given by

$$H = \left[\begin{array}{c|c} I_m & \underbrace{F}_{(m \times k_0)} \\ \hline [0] & \underbrace{H_{\text{MDS}}}_{(\Delta \times k_0)} \end{array} \right],$$

with $H_{\text{MDS}}, F, \underline{x}_i$ as given above and $\Delta \leq (r-1)$, has locality r , the PMR property and minimum distance achieving the bound

$$\begin{aligned} d_{\min} &= (n - k + 1) - \left(\lceil \frac{k}{r} \rceil - 1 \right) \\ &= \Delta + 2. \end{aligned}$$

Proof: We need to show that any $(\Delta+1)$ columns of H are linearly independent. From the properties of the matrix

H_{MDS} , it is not hard to see that it suffices to show that any $(\Delta+1)$ columns of

$$H_a = \left[\frac{F}{H_{\text{MDS}}} \right],$$

are linearly independent. But the rowspace of F contains the vector \underline{x}^t , hence it suffices to show that any $(\Delta+1)$ columns of

$$H_b = \left[\frac{H_{\text{MDS}}}{\underline{x}^t} \right] = H_0$$

are linearly independent, but this is clearly the case, since H_0 is the parity-check matrix of an MDS code having redundancy $(\Delta+1)$. ■

Remark 1: The construction gives rise to codes having parameters $[m(r+1), mr - \Delta, \Delta+2]_r$ and hence, high rate:

$$R = 1 - \frac{\Delta+1}{m(r+1)} \geq 1 - \frac{r}{m(r+1)}.$$

III. A GENERAL APPROACH TO PMR CONSTRUCTION

We attempt to handle the general case

$$\Delta = ar + b,$$

in this section and outline one approach. At this time, we are only able to provide constructions for selected parameters with $\Delta = 2r - 2$ and field size that is cubic in the block length of the code and hold out hope that this construction can be generalized.

The desired minimum distance of the PMR code (with H as given in Theorem 2.3 and H_{MDS} chosen to be a Vandermonde matrix) can be shown to equal in this case,

$$\begin{aligned} d := d_{\min} &= (n - k + 1) - \left(\lceil \frac{k}{r} \rceil - 1 \right) \\ &= (m + \Delta + 1) - \left(\lceil \frac{mr - \Delta}{r} \rceil - 1 \right) \\ &= \Delta + 2 + a. \end{aligned}$$

It follows that even the code on the right having parity-check matrix

$$H_{\text{pun}} = \left[\frac{F}{H_{\text{MDS}}} \right],$$

must have the same value of d_{\min} and therefore, the sub matrix formed by any $(d-1)$ columns of H_{pun} must have full rank. Let A be the support of this subset of $(d-1)$ columns of H_{pun} . Let this support have non-empty intersection with the support of s local codes and the support of the intersection with the i th code being A_i of size $|A_i| = \ell_i$. The corresponding sub matrix will then take on the form:

$$\begin{bmatrix} a_1(\theta_{1i}) & & & & a_2(\theta_{2i}) & & & & \ddots & & & & a_s(\theta_{si}) \\ \vdots & 1 & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \theta_{1i} & \vdots & \vdots & \theta_{2i} & \vdots & \vdots & \vdots & \theta_{si} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \theta_{1i}^2 & \vdots & \vdots & \theta_{2i}^2 & \vdots & \vdots & \vdots & \theta_{si}^2 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \theta_{1i}^{\Delta-1} & \vdots & \vdots & \theta_{2i}^{\Delta-1} & \vdots & \vdots & \vdots & \theta_{si}^{\Delta-1} & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where $a_i(x)$ are the polynomials whose evaluations provide

the local parities. Since we want this matrix to have full rank $(d-1)$ it must be that the left null space of the matrix must be of dimension $(\Delta + s) - (\Delta + a + 1) = s - (a + 1)$. Computing the dimension of this null space is equivalent to computing the number of solutions to

$$\sum_{i=1}^s c_i \sum_{j=1}^{\ell_i} a_i(\theta_{ij}) \prod_{(k,l) \neq (i,j)} \frac{(x - \theta_{kl})}{(\theta_{ij} - \theta_{kl})} = f(x),$$

where $f(x)$ is generic notation for a polynomial of degree $\leq (\Delta - 1)$. Let us define

$$E_i(x) = \sum_{j=1}^{\ell_i} a_i(\theta_{ij}) \prod_{(k,l) \neq (i,j)} \frac{(x - \theta_{kl})}{(\theta_{ij} - \theta_{kl})},$$

and note that each $E_i(x)$ will in general, have degree $(\Delta + a)$. Consider the matrix E whose rows correspond to the coefficients of $E_i(x)$. It follows that the first $(a + 1)$ columns of E must have full rank.

A. Restriction to the Case $a = 1$, i.e., $r \leq \Delta \leq 2r - 1$

We now assume that $a = 1$ so that $(a + 1) = 2$ and we need the first 2 columns of E to have rank $= 2$. We consider the (2×2) sub matrix made up of the first two rows and first two columns of E . The determinant of this (2×2) upper-left matrix formed of E is given by

$$\det \begin{bmatrix} \sum_{j=1}^{\ell_1} \frac{a_1(\theta_{1j})}{P_{1j}} & \sum_{j=1}^{\ell_1} \frac{a_1(\theta_{1j})(\sum_{(k,l) \neq (1,j)} \theta_{kl})}{P_{1j}} \\ \sum_{j=1}^{\ell_2} \frac{a_2(\theta_{2j})}{P_{2j}} & \sum_{j=1}^{\ell_2} \frac{a_2(\theta_{2j})(\sum_{(k,l) \neq (2,j)} \theta_{kl})}{P_{2j}} \end{bmatrix} \\ = -\det \begin{bmatrix} \sum_{j=1}^{\ell_1} \frac{a_1(\theta_{1j})}{P_{1j}} & \sum_{j=1}^{\ell_1} \frac{a_1(\theta_{1j})\theta_{1j}}{P_{1j}} \\ \sum_{j=1}^{\ell_2} \frac{a_2(\theta_{2j})}{P_{2j}} & \sum_{j=1}^{\ell_2} \frac{a_2(\theta_{2j})\theta_{2j}}{P_{2j}} \end{bmatrix}$$

where

$$P_{ij} = \prod_{(k,l) \neq (i,j)} (\theta_{ij} - \theta_{kl})$$

This is equal to

$$\sum_{j=1}^{\ell_1} \sum_{t=1}^{\ell_2} \frac{a_1(\theta_{1j})a_2(\theta_{2t})}{P_{1j}P_{2t}} (\theta_{1j} - \theta_{2t}).$$

Let $\Delta = 2r - 1$ and $a_1(\theta_{1j}) = \theta_{1j}$, $a_2(\theta_{2t}) = \theta_{2t}$, $\theta_{ij} = \xi + h_{ij}$, $h_{ij} \in \mathbb{F}_q$ and $\xi \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$. Then this becomes:

$$\sum_{j=1}^{\ell_1} \sum_{t=1}^{\ell_2} \frac{(\xi^2 + \xi(h_{1j} + h_{2t}) + h_{1j}h_{2t})}{P_{1j}(\theta_{1j})P_{2t}(\theta_{2t})} (\theta_{1j} - \theta_{2t}) \\ = A\xi^2 + B\xi + C$$

with $A, B, C \in \mathbb{F}_q$ which will be nonzero if the minimum polynomial of ξ over \mathbb{F}_q has degree $= 3$, unless all the coefficients are equal to zero.

a) *Numerical Evidence:* Computer verification was carried out for the $\Delta = 5, r = 3$ case for $n = 12$ over $F_{(2^4)^3}$ and $n = 36$ over $F_{(2^6)^3}$ with $h_{ij} = \alpha^{(i-1)}\beta(ij)$ where α is the primitive element of F_{2^4} and F_{2^6} respectively for the two cases and $\beta(ij)$ is fifth and seventh root of unity respectively (the choice of fifth and seventh roots of unity varies for each i, j). For both cases, it was found that the elements A, B, C never simultaneously vanished for all instances.

IV. MAXIMAL RECOVERABLE CODES

A. A Coset-Based Construction with Locality $r = 2$

Since this construction is based on Construction 1 in [9] of all-symbol locality codes, we briefly review the latter here.

Let $n = m(r + 1)$, and q be a power of a prime such that $n \leq (q - 1)$, for example, q could equal $(n + 1)$. Let α be a primitive element of \mathbb{F}_q and β an element of order $(r + 1)$. Let

$$A_i = \alpha^{i-1} \{1, \beta, \beta^2, \dots, \beta^r\}, \quad 1 \leq i \leq m.$$

Note that $\{A_i\}_{i=1}^m$ are pairwise disjoint and partition $[n]$. Let $k = ar + b$. Let the supports of the local codes be $A_i, 1 \leq i \leq m$. Note that the monomial x^{r+1} is constant on each of the sets A_i . Let us set

$$f(x) = \sum_{j=0}^{a-1} \sum_{i=0}^{r-1} a_{ij} x^{j(r+1)+i} + \sum_{j=a}^{b-1} \sum_{i=0}^{r-1} a_{ij} x^{j(r+1)+i},$$

where the second term is vacuous for $b = 0$, i.e., is not present when $r \mid k$. Consider the code \mathcal{C} of block length n and dimension k where each polynomial is associated to a distinct codeword obtained by evaluating the polynomial at the elements of $\bigcup_{i=1}^m A_i$. This code possesses all-symbol locality and has minimum distance d_{\min} satisfying (1).

Note that the exponents e in the monomial terms forming each polynomial $f(x)$ satisfy $e \not\equiv r \pmod{r+1}$. It is this property this property that gives the code its locality properties.

Our construction of an MR code here is based on the above construction with parameters given by $n = q - 1, r = 2, k = 2D + 1$ so that $a = D$ and $b = 1$. Thus the local codes all have length 3. Let us denote the algebraic closure of \mathbb{F}_q by \mathbb{F} .

Theorem 4.1: Given positive integers N, D with $\frac{2D}{N} < \frac{2}{3}$ and

$$q > \Sigma_{j=2}^{2D} \lfloor jg(j) \rfloor \left(\left(\frac{N}{3} - 1 \right) \right) 3^j + N - 2,$$

where

$$g(j) = \begin{cases} 1 & \text{for } j \text{ even and } 2(D-1) \geq j \geq 4 \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

there exists an $[N, k = 2D + 1]$ MR code with $r = 2$ that is obtained from \mathcal{C} by puncturing the code at a carefully selected set of $s = \frac{q-1}{3} - \frac{N}{3}$ cosets $\{A_{i_1}, A_{i_2}, \dots, A_{i_s}\}$.

Proof: Please see the Appendix A. \blacksquare

Example 1: Let $k = 5, n = 15$. The condition in the theorem becomes $q > 499$ whereas, the optimized construction given in [16] requires a field size of 2^{14} . The construction in [10] requires $q > \binom{n-1}{k-1} = 1001$.

B. Modification of Construction by Blaum et al. for $\Delta = 2$

in [15], the authors provide a construction for an MR code (the code is referred to as a partial MDS code in their paper). We present a modification of this construction here. The modification essentially amounts to a different choice of finite-field elements in the construction of the parity check matrix given in [15] for the partial MDS code. The modified parity-check matrix is provided below.

$$H = \begin{pmatrix} H_0 & 0 & \cdots & 0 \\ 0 & H_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_0 \\ H_1 & H_2 & \cdots & H_m \end{pmatrix},$$

where

$$H_j = \begin{pmatrix} 1 & \beta^\delta & \beta^{2\delta} & \cdots & \beta^{(r)\delta} \\ \alpha^{j-1} & \alpha^{j-1}\beta^{-1} & \alpha^{j-1}\beta^{-2} & \cdots & \alpha^{j-1}\beta^{-(r)} \end{pmatrix},$$

and

$$H_0 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \beta^1 & \beta^2 & \cdots & \beta^r \\ 1 & \beta^2 & \beta^4 & \cdots & \beta^{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \beta^{\delta-1} & \beta^{2(\delta-1)} & \cdots & \beta^{r(\delta-1)} \end{pmatrix}.$$

In the above, α is a primitive element of F_q and β is a ψ th root of unity for any $\psi \geq r+1$ and hence ψ divides $q-1$. Using the closed-form expression for the determinant given in [15], it can be seen that this construction yields an MR code with field size $q-1 \geq \psi m$. Note that the field size is independent of δ .

V. NON-EXPLICIT CONSTRUCTION OF MR CODES WITH $O(n^{\Delta-1})$ FIELD SIZE

In this section we provide a construction for MR codes derived by ensuring that certain polynomial constraints which reflect the rank conditions the parity-check matrix of an MR code has to satisfy, hold. Our starting point is the canonical form of the parity-check matrix for an MR code given in Theorem 2.1. In our construction, the sub-matrix H_{MDS} is fixed and we show the existence of assignment of values to the local parities corresponding to the elements of F that result in an MR code. Our approach yields improved field size in comparison with the approach in Lemma 32 of [16].

Theorem 5.1: There exists a choice of x_{ij} such that

$$H = \left[\begin{array}{c|c} I_m & \underbrace{F}_{(m \times k_0)} \\ \hline [0] & \underbrace{H_{MDS}}_{(\Delta \times k_0)} \end{array} \right],$$

$$F = \begin{bmatrix} \underline{x}_1^t & & & \\ & \underline{x}_2^t & & \\ & & \ddots & \\ & & & \underline{x}_m^t \end{bmatrix}$$

$$\underline{x}_i^t = (x_{i1}, x_{i1}, \dots, x_{ir})$$

is a maximally recoverable code for any H_{MDS} with a field size of $O(n^{\Delta-1})$ (for fixed r, Δ).

Proof: The proof is skipped for lack of space. ■

The above construction can be extended in a straight forward manner to give maximal recoverable codes with field size of $O(n^{\Delta-1})$ when the matrix F is made up of blocks of $\delta \times (r+1)$ local codes where we correct δ erasures in each local code.

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APPENDIX A

PROOFS OF THEOREMS ON MAXIMAL RECOVERABILITY

Proof of Theorem 4.1: The code C has optimum minimum distance w.r.t locality $r = 2$ [1]. Hence puncturing at any number of cosets (local codes) without changing k will maintain the optimum minimum distance. We say that e is an admissible puncturing pattern if $e \subset [N]$ and $|e \cap L_i| = 1$, all i .

Let F be the algebraic closure of F_q . Throughout the proof whenever we say a pattern e or just e , it refers to an admissible puncturing pattern for an $[N, k]$ code with all symbol locality r . Throughout the discussion any $[N, k]$ code referred to are polynomial evaluation codes and we assume that the set of evaluation positions of the $[N, k]$ code to be ordered. We use e also to indicate the actual finite field elements at the positions indicated by the puncturing pattern e in the set of evaluation positions of the $[N, k]$ code.

Maximal Recoverability:

Let $l = \frac{N}{3}$.

We denote an encoding polynomial of C by $f(x)$ and we assume $f \neq 0$. Let H denote the cyclic group of cube roots of unity. Let α be a primitive element in F_q . If $\{X_1, \dots, X_{3D}\} \subset F$ are the roots of $f(x)$ then it must satisfy:

$$\begin{aligned} \sigma_1(X_1, \dots, X_{3D}) &= 0 \\ \sigma_4(X_1, \dots, X_{3D}) &= 0 \\ &\vdots \\ \sigma_{1+3(D-1)}(X_1, \dots, X_{3D}) &= 0 \end{aligned}$$

where σ_i refers to the i th elementary symmetric function. Lets denote the above set of conditions based on elementary symmetric functions on X_1, \dots, X_{3D} by $R(D)$.

If we have a $[N, k = 2D + 1]$ maximally recoverable code based on the theorem and let H_1, \dots, H_l be the chosen cosets of evaluation positions for forming the codeword of the $[N, k]$ maximally recoverable code and if we puncture this $[N, k]$ code by a pattern e then for the resulting $[N - l, k]$ (assuming k doesn't change after puncturing) code to be MDS we need $d_{\min} = N - l - k + 1 = N - 2D - l$. Based on the degree of $f(x)$, we know that $d_{\min} \geq N - l - \deg(f) = N - l - 3D$. Hence out of $3D$ roots of $f(x)$, we want atleast D roots to lie outside $H_1 - e(1), \dots, H_l - e(l)$ for any e . In other words its enough if we choose l cosets such that for any $\{X_1, \dots, X_{3D}\} \subset F$ which satisfies the condition $R(D)$, atleast only $2D$ distinct elements will lie in the chosen l cosets after puncturing by any e . Note that this condition will also ensure that the dimension of a $N - l$ length punctured code obtained by puncturing the $[N, k]$ code by a pattern e is k for any e . If not there are 2 distinct non zero message polynomials $f_1(x), f_2(x)$ which after evaluating at l cosets of evaluation positions of the $[N, k]$ code yields the same codeword after puncturing by a pattern e to $N - l$ length. This means $f_1 - f_2$ is another non zero message or evaluation polynomial with $N - l$ zeros in the chosen l cosets after puncturing by e but by the condition of choosing cosets mentioned in

previous sentence (roots of $f_1 - f_2$ satisfies $R(D)$) there can be atleast $2D$ distinct zeros in the $N - l$ evaluation positions. This is a contradiction as $N - l = \frac{2N}{3} > 2D$ (by the condition $\frac{2D}{N} < \frac{2}{3}$ given in the theorem). Hence if we choose l cosets such that for any pattern e and any $2D$ distinct elements X_1, \dots, X_{2D} from the l cosets after puncturing by e , none of X_{2D+1}, \dots, X_{3D} from F such that X_1, \dots, X_{3D} satisfies $R(D)$ which are distinct from X_1, \dots, X_{2D} lie in the chosen cosets after puncturing by e then we are done.

Proposition 1: Let S be a set of elements $3A$ elements from F satisfying $R(A)$ and S contains $\alpha^i H$ for some i then $S - \alpha^i H$ satisfies $R(A - 1)$.

Proof: Since S satisfies $R(A)$, this implies $\sigma_{1+3(i-1)}(S) = 0$ for $i = 1, \dots, A$.

$$\begin{aligned} \sigma_{1+3(i-1)}(S) &= \sigma_3(\alpha^i H) \sigma_{1+3(i-1)-3}(S - \alpha^i H) + \\ &\quad \sigma_2(\alpha^i H) \sigma_{1+3(i-1)-2}(S - \alpha^i H) + \\ &\quad \sigma_1(\alpha^i H) \sigma_{1+3(i-1)-1}(S - \alpha^i H) + \sigma_{1+3(i-1)}(S - \alpha^i H) \end{aligned}$$

$$\sigma_3(\alpha^i H) = a, \sigma_2(\alpha^i H) = 0, \sigma_1(\alpha^i H) = 0, \quad \text{for some } a \neq 0.$$

Hence

$$\sigma_{1+3(i-1)}(S) = a \sigma_{1+3(i-1)-3}(S - \alpha^i H) + \sigma_{1+3(i-1)}(S - \alpha^i H)$$

For $i = A$, $\sigma_{1+3(A-1)}(S - \alpha^i H) = 0$ as $S - \alpha^i H$ has only $3(A - 1)$ elements.

Hence,

$$\sigma_{1+3(A-1)}(S) = a \sigma_{1+3(A-1)-3}(S - \alpha^i H)$$

Hence

$$\sigma_{1+3(A-1)}(S) = 0 \Rightarrow \sigma_{1+3(A-1)-3}(S - \alpha^i H) = 0$$

for $i = A - 1$,

$$\begin{aligned} \sigma_{1+3(A-2)}(S) &= a \sigma_{1+3(A-2)-3}(S - \alpha^i H) + \\ &\quad \sigma_{1+3(A-2)}(S - \alpha^i H) \end{aligned}$$

Since, $\sigma_{1+3(A-2)}(S) = 0$ and $\sigma_{1+3(A-2)}(S - \alpha^i H) = 0$, this implies that $\sigma_{1+3(A-2)-3}(S - \alpha^i H) = 0$

By induction, if we assume, $\sigma_{1+3(i-1)}(S - \alpha^i H) = 0$ then since $\sigma_{1+3(i-1)}(S) = 0$, we have

$\sigma_{1+3(i-1)-3}(S - \alpha^i H) = 0$ ($i = A$ is the starting condition of the induction which we already proved).

Hence $S - \alpha^i H$ satisfies $R(A - 1)$. ■

Claim:

Its enough to choose l cosets such that for any $(A \leq D)$ and any X_1, \dots, X_{2A} (contained in the chosen l cosets) which are distinct and contains atleast one element from each coset, none of the X_{2A+1}, \dots, X_{3A} from F such that X_1, \dots, X_{3A} satisfies $R(A)$, which are distinct from X_1, \dots, X_{2A} lies in the chosen l cosets after puncturing by e for any e disjoint from X_1, \dots, X_{2A} .

Proof:

This is because if X_1, \dots, X_{3D} satisfying $R(D)$ contains at least 2 element from some coset $\alpha^i H$ for some i , since the polynomial $f_1(x) = (x - X_1) \dots (x - X_{3D})$ restricted to any coset is a degree 1 polynomial, the third element from coset is also a root of f_1 . Hence the entire coset is contained in X_1, \dots, X_{3D} and by similar reasoning X_1, \dots, X_{3D} can be written as $X_1, \dots, X_{3(D-j)} \cup \alpha^{i_1} H \cup \dots \alpha^{i_j} H$ for some i_1, \dots, i_j where $X_1, \dots, X_{3(D-j)}$ contains at most one element from each coset and satisfies $R(D-j)$ by proposition 1.

Now by the property of the chosen cosets, we have that for any distinct $X_1, \dots, X_{2(D-j)}$ from the chosen l cosets containing atmost one element from each coset, any of $X_{2(D-j)+1}, \dots, X_{3(D-j)}$ which are distinct from $X_1, \dots, X_{2(D-j)}$ such that $X_1, \dots, X_{3(D-j)}$ satisfies $R(D-j)$ will not lie inside the chosen cosets after puncturing by e for any e such that $e \cap \{X_1, \dots, X_{2(D-j)}\} = \emptyset$. Wlog this implies the chosen cosets after puncturing by any e can contain atmost only (writing only distinct elements) $X_1, \dots, X_{2(D-j)} \cup \alpha^{i_1} H - e(i_1) \cup \dots \alpha^{i_j} H - e(i_j)$ of the $3D$ elements. Hence there can be atmost $2(D-j) + 3j - j = 2D$ roots out of $3D$ roots inside the chosen cosets after puncturing by any e . Hence we are done.

From here we term a set of l cosets satisfying the above claim, to be satisfying $R_1(l)$.

We are going put another set of conditions $R_2(l)$ on a set of l cosets. The necessity of this condition will be clear in the proof.

$R_2(l)$:

A given set of l cosets, is said to satisfy condition $R_2(l)$ if, For any $1 \leq A \leq D$ and any X_1, \dots, X_{2A} (contained in the chosen l cosets) which are distinct and contains atmost one element from each of l cosets, the matrix $P(A)$ given by

$P(A) =$

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S) & \sigma_2(S) & \dots & \dots & \sigma_{3+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S) & \sigma_{3(i-1)-1}(S) & \dots & \dots & \sigma_{3(i-1)+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{2A}(S) & \sigma_{2A-1}(S) & \sigma_{2(A-1)}(S) \end{pmatrix}$$

is non-singular, where $S = \{X_1, \dots, X_{2A}\}$.

Furthermore, for any $3 \leq A \leq D$ and any X_1, \dots, X_{2A-1} (contained in the chosen l cosets) which are distinct and contains at most one element from each of l cosets, the matrix $P_1(A)$ given by

$P_1(A) =$

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S) & \sigma_2(S) & \dots & \dots & \sigma_{3+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S) & \sigma_{3(i-1)-1}(S) & \dots & \dots & \sigma_{3(i-1)+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_{2A-1}(S) & \sigma_{2(A-1)}(S) \end{pmatrix}$$

is non-singular. where $S = \{X_1, \dots, X_{2A-1}\}$.

From here on we proceed to find a set of l cosets satisfying $R_1(l)$ and $R_2(l)$. We proceed by choosing 1 coset at each step inductively until we choose the required set of l cosets. At each step we select and add one coset to our list and throw away a collection of cosets from the cosets not chosen. Let the cosets chosen upto i th step be $G(i)$ and the cosets thrown upto i th step be $T(i)$ and let the total collection of cosets in the field F_q be W .

1) The first coset is chosen to be any coset. Hence $G(1)$ consists of just the coset chosen. We don't throw away any cosets at this step. Hence $T(1)$ is empty. $G(1)$ satisfies $R_1(1)$ and $R_2(1)$ trivially.

2) The second coset is also chosen to be any coset from $W - G(1)$. Hence $G(2)$ consists of the 2 chosen cosets. $R_1(2)$:

For $A = 1$, and for any $2A = 2$ distinct elements X_1, X_2 , one from each coset in $G(2)$, any X_3 such that $\sigma_1(X_1, X_2, X_3) = 0$ cannot be distinct from X_1, X_2 and lie in any of the cosets in $G(2)$. If it does, wlog let X_1 and X_3 lie in same coset which is in $G(2)$ then $X_2 = -(X_1 + X_3)$ but every coset is a coset of cube roots of unity. Hence $X + X_1 + X_3 = 0$ where X is the third element from the same coset as X_1, X_3 . Hence $X = -(X_1 + X_3)$ which implies $X = X_2$ but X is in the same coset as X_1, X_3 and X_2 is in the other coset in $G(2)$. Hence a contradiction.

This implies that additive inverse of sum of 2 distinct elements from different cosets cannot lie in the same coset as the 2 elements.

For $A \geq 2$, $2A \geq 4$, we need to pick 4 distinct elements, from distinct cosets but there are only 2 cosets in $G(2)$. Hence $R_1(2)$ is satisfied.

$R_2(2)$:

For $A = 1$, $P_1(1) = [1]$, $P_2(1) = [1]$, hence non-singular.

For $A \geq 2$, we need to pick $2A \geq 4$ and $2A - 1 \geq 3$ distinct elements from distinct cosets but there are only 2 cosets. Hence $R_2(2)$ is satisfied.

$T(2)$:

For every two distinct elements X_1, X_2 chosen one from each of the 2 cosets in $G(2)$, find the third element X_3 such that $\sigma_1(X_1, X_2, X_3) = 0$ and throw away the coset in $W - (G(2))$ which contains it. Since $G(2)$ satisfies $R_1(2)$, X_3 will either not lie any coset in $G(2)$ or won't be distinct from X_1, X_2 . In the first case, we throw the coset and in the latter case, we don't do anything. There are $3 \times 3 = 9$ possible summations $X_1 + X_2$ but if $X_1 + X_2 + X_3 = 0$ then $\theta(X_1 + X_2 + X_3) = 0$ and θX_3 is in the same coset as X_3 for any cube root of unity θ . Hence solutions for 9 possible summations lie in atmost 3 cosets and we throw away these 3 cosets.

3) Let $i \geq 2D$ and assume we have $G(i)$ satisfying $R_1(i), R_2(i)$.

$T(i)$:

a) For every $A \leq D$, Choose $2A$ cosets (say H_1, \dots, H_{2A}) out of $G(i)$ cosets, and choose X_1, \dots, X_{2A} one from each of these $2A$ cosets, now find the set of all X_{2A+1}, \dots, X_{3A} from F such that $\sigma_1(X_1, \dots, X_{3A}) = 0, \dots, \sigma_{1+3(A-1)}(X_1, \dots, X_{3A}) = 0$ and throw away all the cosets in which X_{2A+1}, \dots, X_{3A} lies. Since $G(i)$ satisfies $R_1(i)$, the elements in X_{2A+1}, \dots, X_{3A} will either be not distinct from X_1, \dots, X_{2A} or will lie outside $G(i)$. In the first case we don't do anything and in the latter case, we throw away these cosets.

To find the number of solutions X_{2A+1}, \dots, X_{3A} such that $\sigma_1(X_1, \dots, X_{3A}) = 0, \dots, \sigma_{1+3(A-1)}(X_1, \dots, X_{3A}) = 0$, we solve for X_{2A+1}, \dots, X_{3A} given X_1, \dots, X_{2A} .

It can be seen that to satisfy $\sigma_1(X_1, \dots, X_{3A}) = 0, \dots, \sigma_{1+3(A-1)}(X_1, \dots, X_{3A}) = 0$, $\sigma_1(X_{2A+1}, \dots, X_{3A}), \dots, \sigma_A(X_{2A+1}, \dots, X_{3A})$ has to satisfy a linear equation of the form

$$P(A)[\sigma_1(X_{2A+1}, \dots, X_{3A}), \dots, \sigma_A(X_{2A+1}, \dots, X_{3A})]^t = -[\sigma_1(X_1, \dots, X_{2A}), \dots, \sigma_{1+3(A-1)}(X_{2A+1}, \dots, X_{3A})]^t$$

since $G(i)$ satisfies $R_2(i)$, $P(A)$ is non singular and there is a unique solution, for

$[\sigma_1(X_{2A+1}, \dots, X_{3A}), \dots, \sigma_A(X_{2A+1}, \dots, X_{3A})]$ which implies a unique solution for X_{2A+1}, \dots, X_{3A} . Hence for a given distinct X_1, \dots, X_{2A} , from distinct cosets, there is a unique solution for X_{2A+1}, \dots, X_{3A} such that $\sigma_1(X_1, \dots, X_{3A}) = 0, \dots, \sigma_{1+3(A-1)}(X_1, \dots, X_{3A}) = 0$. Hence its enough to throw these A cosets containing X_{2A+1}, \dots, X_{3A} (unique solution).

The above procedure is done for every choice of $2A$ cosets from $G(i)$ cosets and every choice of X_1, \dots, X_{2A} from the chosen $2A$ cosets.

Hence the total number of cosets thrown are atmost $(|G(i)|) 3^{2A} A$ but if for X_1, \dots, X_{2A} , X_{2A+1}, \dots, X_{3A} put together satisfies $R(A)$ then for $\theta(X_1, \dots, X_{2A})$, $\theta(X_{2A+1}, \dots, X_{3A})$ (which doesn't change the cosets of X_{2A+1}, \dots, X_{3A} for any cube root of unity θ) satisfies $R(A)$ and this choice is unique as seen before. Hence out of 3^{2A} choices for X_1, \dots, X_{2A} from a given chosen $2A$ cosets, its enough to throw away cosets for $\frac{3^{2A}}{3}$ choices of X_1, \dots, X_{2A} .

Hence the total number of cosets thrown are atmost $(|G(i)|) 3^{2A-1} A$.

b) For every $3 \leq A \leq D$, Choose $2A - 1$ cosets (say H_1, \dots, H_{2A-1}) out of $G(i)$ cosets, and choose X_1, \dots, X_{2A-1} one from each of these $2A - 1$ cosets, now find the set of all X_{2A} from F such that $P(A)$ is singular. This X_{2A} can't be in any coset in $G(i)$ which doesn't contain X_1, \dots, X_{2A-1} as $G(i)$ satisfies $R_2(i)$. If X_{2A} lies in the coset which contains any of X_1, \dots, X_{2A-1} , then we don't do anything. If X_{2A} lies outside $G(i)$, we throw the coset. To find the number of solutions of X_4 for a given X_1, \dots, X_3 such that $P(A)$ is singular,

let $S_1 = \{X_1, \dots, X_{2A-1}\}$ and $S = \{X_1, \dots, X_{2A-1}, X_{2A}\}$. $P(A) =$

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S) & \sigma_2(S) & \dots & \dots & \sigma_{3+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S) & \sigma_{3(i-1)-1}(S) & \dots & \dots & \sigma_{3(i-1)+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{2A}(S) & \sigma_{2A-1}(S) & \sigma_{2(A-1)}(S) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_2(S_1) + \sigma_2(S_1)X_{2A} & \sigma_2(S_1) + \sigma_1(S_1)X_{2A} & \dots & \dots & \sigma_{3+1-A}(S_1) + \sigma_{3+1-A-1}(S_1)X_{2A} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S_1) + \sigma_{3(i-1)-1}(S_1)X_{2A} & \sigma_{3(i-1)-1}(S_1) + \sigma_{3(i-1)-2}(S_1)X_{2A} & \dots & \dots & \sigma_{3(i-1)+1-A}(S_1) + \sigma_{3(i-1)+1-A-1}(S_1)X_{2A} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{2A-1}(S_1)X_{2A} & \sigma_{2A-1}(S_1) + \sigma_{2A-2}(S_1)X_{2A} & \sigma_{2(A-1)}(S_1) + \sigma_{2(A-1)-1}(S_1)X_{2A} \end{pmatrix}$$

The determinant of above matrix $P(A)$ can be seen as a polynomial in X_{2A} and its degree is atmost $A - 1$. The constant term of this polynomial is the determinant of following matrix:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S_1) & \sigma_2(S_1) & \dots & \dots & \sigma_{3+1-A}(S_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S_1) & \sigma_{3(i-1)-1}(S_1) & \dots & \dots & \sigma_{3(i-1)+1-A}(S_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_{2A-1}(S_1) & \sigma_{2(A-1)}(S_1) \end{pmatrix}$$

The above matrix is non-singular for $A \geq 3$ since $G(i)$ satisfies $R_2(i)$. Hence the $\det(P(A))$ as a polynomial in X_{2A} is a non-zero polynomial (has a non zero constant term), and since its degree is atmost $A - 1$, it can have atmost $A - 1$ solutions for X_{2A} . Hence its enough to throw away these $A - 1$ cosets containing these $A - 1$ solutions.

The above procedure is done for every choice of $2A - 1$ cosets from $G(i)$ cosets and every choice of X_1, \dots, X_{2A-1} from the chosen $2A - 1$ cosets.

The number of cosets thrown are atmost: $(|G(i)|) 3^{2A-1} (A - 1)$. It can be seen that $\det(P(A))$ is a homogenous polynomial in X_1, \dots, X_{2A} and hence as before if for $X_1, \dots, X_{2A-1}, X_{2A}$, $\det(P(A)) = 0$ then for $\theta(X_1, \dots, X_{2A-1}, X_{2A})$ also $\det(P(A)) = 0$. Hence its enough to throw away atmost: $(|G(i)|) 3^{2A-2} (A - 1)$.

For $A = 1$, $P(A) = [1]$ which is trivially non-singular and we don't do anything. For $A = 2$, choose 3 cosets from $G(i)$ and choose distinct X_1, X_2, X_3 one from each of these distinct cosets, now find the set of all X_4 such that $P(A)$ is singular. This X_4 can't be in any coset in $G(i)$ which doesn't contain X_1, \dots, X_3 as $G(i)$ satisfies $R_2(i)$. If X_4 lies in the coset which contains any of X_1, \dots, X_3 , then we don't do anything. If X_4 lies outside $G(i)$, we throw the coset. To find the number of solutions of X_4 for a given X_1, \dots, X_3 such that $P(A)$ is singular,

$$\det(P(A)) = \sigma_2(X_1, X_2, X_3, X_4) = \sigma_2(X_1, X_2, X_3) + X_4 \sigma_1(X_1, X_2, X_3)$$

Given the chosen X_1, X_2, X_3 , the above expression for $\det(P(A))$ can be seen as a linear expression in X_4 . If

$\sigma_2(X_1, X_2, X_3) = 0, \sigma_1(X_1, X_2, X_3) = 0$ then $(X - X_1)(X - X_2)(X - X_3) = X^3 - X_1X_2X_3 = X^3 - \gamma$. Here X_1, X_2, X_3 constitutes the solution set for $X^3 = \gamma$ but X_1H also constitutes 3 solutions for the equation $X^3 = \gamma$ but there can be atmost 3 solutions for the equation $X^3 = \gamma$. Hence $X_1H = \{X_1, X_2, X_3\}$ which implies they all belong to same coset which is a contradiction. Hence either $\sigma_2(X_1, X_2, X_3) \neq 0$ or $\sigma_1(X_1, X_2, X_3) \neq 0$ which implies $\det(P(A))$ is a non zero degree 1 polynomial in X_4 . Hence we can find the solution and throw away the coset containing it.

The number of cosets thrown are atmost: $\binom{|G(i)|}{3}3^3$ but by similar argument as before we can see that the number of cosets thrown are atmost: $\binom{|G(i)|}{3}3^2$

c) For every $3 \leq A \leq D$, Choose $2A - 2$ cosets (say H_1, \dots, H_{2A-2}) out of $G(i)$ cosets, and choose X_1, \dots, X_{2A-2} one from each of these $2A - 2$ cosets, now find the set of all X_{2A-1} from F such that $P_1(A)$ is singular. This X_{2A-1} can't be in any coset in $G(i)$ which doesn't contain X_1, \dots, X_{2A-2} as $G(i)$ satisfies $R_2(i)$. If X_{2A-1} lies in the coset which contains any of X_1, \dots, X_{2A-2} , then we don't do anything. If X_{2A-1} lies outside $G(i)$, we throw the coset.

To find the number of solutions of X_{2A-1} for a given X_1, \dots, X_{2A-2} such that $P_1(A)$ singular, let $S_1 = \{X_1, \dots, X_{2A-2}\}$ and $S = \{X_1, \dots, X_{2A-2}, X_{2A-1}\}$.

$$P_1(A) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S) & \sigma_2(S) & \dots & \dots & \sigma_{3+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S) & \sigma_{3(i-1)-1}(S) & \dots & \dots & \sigma_{3(i-1)+1-A}(S) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_{2A-1}(S) & \sigma_{2(A-1)}(S) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S_1) + \sigma_3(S_i)X_{2A-1} & \sigma_2(S_1) + \sigma_2(S_i)X_{2A-1} & \dots & \dots & \sigma_{3+1-A}(S_1) + \sigma_{3+1-A}(S_i)X_{2A-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S_1) + \sigma_{3(i-1)-1}(S_i)X_{2A-1} & \sigma_{3(i-1)-1}(S_1) + \sigma_{3(i-1)-1}(S_i)X_{2A-1} & \dots & \dots & \sigma_{3(i-1)+1-A}(S_1) + \sigma_{3(i-1)+1-A}(S_i)X_{2A-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_{2A-1}(S_1) + \sigma_{2A-1}(S_i)X_{2A-1} & \sigma_{2(A-1)}(S_1) + \sigma_{2(A-1)}(S_i)X_{2A-1} \end{pmatrix}$$

The determinant of above matrix $P_1(A)$ can be seen as a polynomial in X_{2A-1} and its degree is atmost $A - 1$. The constant term of this polynomial is the determinant of following matrix:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S_1) & \sigma_2(S_1) & \dots & \dots & \sigma_{3+1-A}(S_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S_1) & \sigma_{3(i-1)-1}(S_1) & \dots & \dots & \sigma_{3(i-1)+1-A}(S_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \sigma_{2(A-1)}(S_1) \end{pmatrix}$$

Now $\sigma_{2(A-1)}(S_1) \neq 0$ (because this is just the product of $2(A - 1)$ non zero elements), since the determinant of the matrix:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \sigma_3(S_1) & \sigma_2(S_1) & \dots & \dots & \sigma_{3+1-A}(S_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{3(i-1)}(S_1) & \sigma_{3(i-1)-1}(S_1) & \dots & \dots & \sigma_{3(i-1)+1-A}(S_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{2(A-1)}(S_1) & \sigma_{2(A-1)-1}(S_1) & \sigma_{2(A-2)}(S_1) \end{pmatrix}$$

is non zero (because $G(i)$ satisfies $R_2(i)$), we have that the determinant of the matrix mentioned before corresponding to the constant term of the polynomial $\det(P_1(A))$ is also non zero. Hence the $\det(P_1(A))$ as a polynomial in X_{2A-1} is a non-zero polynomial (has a non zero constant term), and since its degree is atmost $A - 1$, it can have atmost $A - 1$ solutions for X_{2A-1} . Hence its enough to throw away these $A - 1$ cosets containing these $A - 1$ solutions.

The above procedure is done for every choice of $2A - 2$ cosets from $G(i)$ cosets and every choice of X_1, \dots, X_{2A-2} from the chosen $2A - 2$ cosets.

The number of cosets thrown are atmost: $\binom{|G(i)|}{2A-2}3^{2A-2}(A - 1)$. It can be seen that $\det(P_1(A))$ is a homogenous polynomial in X_1, \dots, X_{2A-1} and hence as before if for $X_1, \dots, X_{2A-2}, X_{2A-1}$, $\det(P_1(A)) = 0$ then for $\theta(X_1, \dots, X_{2A-2}, X_{2A-1})$ also $\det(P_1(A)) = 0$. Hence its enough to throw away atmost: $\binom{|G(i)|}{2A-2}3^{2A-3}(A - 1)$.

4) Following the previous step, we want to select one more coset to form $G(i+1)$ such that it satisfies $R_1(i+1), R_2(i+1)$:

Choose any coset (say H_1 from the collection $W - (T(i) \cup G(i))$. Hence $G(i+1) = G(i) \cup \{H_1\}$. It can be easily shown that $G(i+1)$ satisfies $R_1(i+1), R_2(i+1)$ using the properties of $T(i)$ and $G(i)$. We skip the proof due to space constraints.

5) The argument for throwing cosets for $i < 2D$ is similar to the above arguments (point 3) except that we skip the parts where it becomes vacuous. The procedure for selecting new coset to form $G(i)$ and showing that it satisfies $R_1(i)$ and $R_2(i)$ can be done in a straight forward manner.

We repeat the steps 3 and 4 until we pick l cosets. Note that the set of cosets thrown away at i th step contains the set of cosets thrown away at $i - 1$ th step.

Hence the total number of cosets thrown until i th step is (from the step 3):

$$|T(i)| \leq \sum_{j=1}^D \binom{|G(i)|}{2j} 3^{2j-1}j + \sum_{j=3}^D \binom{|G(i)|}{2j-1} 3^{2j-2}(j-1) + \binom{|G(i)|}{3} 3^2 + \sum_{j=3}^D \binom{|G(i)|}{2j-2} 3^{2j-3}(j-1)$$

we can pick $i+1$ th coset to form $G(i+1)$ as long as $|T(i)| + |G(i)| < |W|$. Hence we can pick $l = \frac{n}{3}$ cosets (evaluating positions) to form maximally recoverable code of block length n as long as $|T(l-1)| + |G(l-1)| < |W|$. $|W| = \frac{q-1}{3}$. Hence we can form $[n, k = 2D + 1]$ maximally recoverable code as long as:

$$\sum_{j=1}^D \binom{|G(l-1)|}{2j} 3^{2j-1}j + \sum_{j=3}^D \binom{|G(l-1)|}{2j-1} 3^{2j-2}(j-1) + \binom{|G(l-1)|}{3} 3^2 + \sum_{j=3}^D \binom{|G(l-1)|}{2j-2} 3^{2j-3}(j-1) + |G(l-1)| < \frac{q-1}{3}$$

Using $|G(i)| = i$, it can be seen that the above inequality is implied by:

$$\Sigma_{j=2}^{2D} \lfloor jg(j) \rfloor \binom{l-1}{j} 3^{j-1} + (l-1) < \frac{q-1}{3}$$

$$g(j) = 1 \text{ for } 2(D-1) \geq j \geq 4 \text{ and } j \text{ even}$$

$$g(j) = \frac{1}{2} \text{ otherwise}$$

hence:

$$\Sigma_{j=2}^{2D} \lfloor jg(j) \rfloor \binom{\left(\frac{n}{3}-1\right)}{j} 3^j + n - 2 < q$$

■